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in

Mechanical Engineering News

28(3):9-13

PICO
Polar Ice Coring Office

Polar Ice Coring Office
University of Alaska Fairbanks
Fairbanks, Alaska 99775-1710

PICO
TJC-103

July 1992

PICO is operated by the University of Alaska Fairbanks under contract to the National Science Foundation, Division of Polar Programs.

MECHANICAL ENGINEERING NEWS

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Mechanical Engineering News (USPS 399-510, ISSN 0025-651X) is published quarterly for $9.00 per year for departmental and individual subscribers and for $15.00 per year for libraries and international subscribers by the Mechanical Engineering Division of the American Society for Engineering Education, California Polytechnic State University, San Luis Obispo, San Luis Obispo, CA 93407. Second-Class postage

Analytical Solutions for Determining Ice Core Temperatures

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Abstract

Analytical solutions have been developed in cylindrical coordinates to predict temperature profiles in ice cores. Three geometric configurations have been investigated: (1) an infinite cylinder; (2) a semi-infinite cylinder; and (3) a finite cylinder. Final solutions have been compactly listed and results generated from them are presented as nondimensional temperature charts which would be useful to the students currently engaged in the design and development of a thermo-mechanical ice coring device at our university.

Introduction

The Polar Ice Coring Office (PICO), operated by the University of Alaska Fairbanks (UAF) for the National Science Foundation, is charged with the development and operation of ice coring drills and augers for scientific research. The primary goal is to sample deep within the world's ice caps to study evidence of past climatic variations and global warming.

Common to all of the deep ice sampling devices to date has been the use of thousands of gallons of drilling fluids, such as diesel fuels, trichloroethylene, fluorocarbons, etc., in a designated pristine environment. Due to environmental concerns, an alternative deep ice coring thermo-mechanical drill which would be both environmentally safe and effective is being developed at the Mechanical Engineering Department of UAF.

The ice coring program has lead to a number of educational projects for our mechanical engineering students at both graduate and undergraduate levels. At the undergraduate level, engineering design plays a central role and individual students have participated in building components of different drill systems and testing them in the field. At the graduate level, independent research has been undertaken on heat transfer modeling of the drill and the ice core. Future plans call for experimental studies at the ice test well completed by UAF in 1990. The present article pertains to heat transfer education and shows how simple analytical methods developed herein can be applied to practical engineering problems.

The internal flow geometry of the hot water coring drill is shown schematically in Figure 1. It involves hot water flow in the outer annulus that melts the ice near the tip of cutting tools and assists in drilling at a faster rate. The cooler water flowing through the inner annulus carries the chips up and eventually melts them. The water is circulated via hoses to the top of the hole where the cooler water goes through a heater and is injected back through the hot water hose.

This is a new technology. Rinaldi *et al.* (1990) have reported that, although hot water drilling has been successful, a coring device of this type has not been developed. A very important factor in this design is to use an extremely low thermal conductivity material for the innermost cylinder surrounding

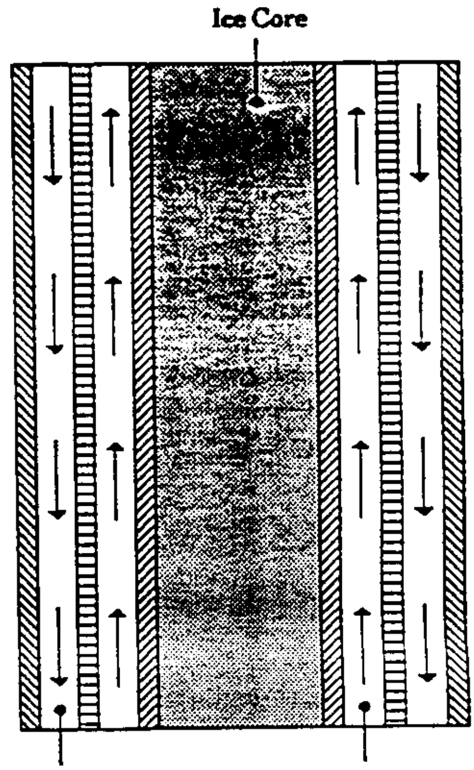

Figure 1. Internal flow geometry of the thermo-mechanical coring drill.

the core. Before we can design a prototype hot water-mechanical drill, we must have an estimate on the rate of warming of ice cores via analytical heat transfer studies. The present study constitutes the basis for analyzing the heat transmission into the ice core before the design and field test of such a drilling device.

During the first phase we have focused on one fundamental problem: to determine the interior temperature distribution in a cylindrical ice core subjected to different temperatures at the boundary due to the circulation of hot water. Treating the core in three ways: (1) an infinite cylinder; (2) a semi-infinite cylinder; and (3) a finite cylinder, we have developed methods to predict how temperature distributions are changing with time. We wish to find out how fast the core temperature rises, and what the risk is of melting the core. The three methods are described in the next section. Using them one can determine the limits on minimum core diameter and the period of drilling that is permissible through thermal coring techniques without jeopardizing the interior region of the core due to excessive penetration of heat.

Analytical Methods

Infinite Cylinder Approach

Assuming constant properties, no heat generation and no dependence on axial and circumferential direction, the appropriate form of the heat conduction equation can be written from Ozisik (1980).

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

The boundary and initial conditions are

$T(b,t) = T_s$, the surface temperature;

$T(r,o) = T_o$, the initial temperature (2)

Let us nondimensionalize the governing equation (1) and boundary and initial conditions (2) using the following dimensionless variables.

$$\Theta_1 = \frac{T_s - T}{T_s - T_o};\ R = \frac{r}{b};\ \tau = \frac{\alpha t}{b^2}$$

$$\tau = \frac{\text{dimensionless time}}{\text{or Fourier number}} = F_o \tag{3}$$

Here Θ_1 is dimensionless temperature, R the dimensionless radial location, b the ice core radius, τ the time, and α the thermal diffusivity of ice.

Following the method outlined by Ozisik (1980) the solution to equations (1) and (2) were derived in Das *et al.* (1991) with the final result:

$$\Theta_1(R,\tau) = 2\sum_{m=1}^{\infty} e^{-\beta_m^2 \tau} \frac{J_0(\beta_m R)}{\beta_m J_1(\beta_m)} \tag{4}$$

where β_m's are positive roots of the Bessel function $J_0(\beta_m) = 0$. The first ten roots of the Bessel function are listed in White (1974). For roots greater than ten, an equation is given in White. We have incorporated these roots and the equation in a comprehensive computer program which is listed in Appendix 1 of Das *et al.* (1991).

Semi-infinite Cylinder Approach

This is a two-dimensional problem with the governing equation given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5}$$

The solution of this type of problem can be written as the product of two one-dimensional solutions, assuming the problem to be linear and homogeneous (Myers, 1987). The product solution is comprised of a semi-infinite body and the infinite cylinder. The governing conduction equation for a semi-infinite solid in the nondimensional form is

$$\frac{\partial^2 \Theta_2}{\partial Z^2} = \frac{\partial \Theta_2}{\partial \tau} \tag{6}$$

Its solution can be written as

$$\Theta_2(Z,\tau) = \text{erf}\left(\frac{Z}{\sqrt{4\tau}}\right) \tag{7}$$

These formulae have been derived in detail in Das *et al.* (1991).

The final result for the semi-infinite cylinder problem appears as:

$$\Theta_{sc}(R,Z,\tau) = \text{erf}\left(\frac{Z}{\sqrt{4\tau}}\right)\left[2\sum_{m=1}^{\infty} e^{-\beta_m^2 \tau} \frac{J_0(\beta_m R)}{\beta_m J_1(\beta_m)}\right] \tag{8}$$

Here Θ_{sc} is the dimensionless temperature in the semi-infinite cylinder and $Z = z/b$ is the dimensionless axial location in the ice core. The computer program to calculate temperatures combining the infinite cylinder program with an error function program is listed in Appendix 2 of Das *et al.* (1991).

Finite Cylinder Approach

The governing equation for this case is the same as equation (5). The solution for this case can also be obtained as the product of two one-dimensional solutions. The product solution is dependent upon heat conduction equations for a slab whose thickness is equal to the height of the finite cylinder and that of an infinite cylinder. The dimensionless conduction equation for a slab is

$$\frac{\partial^2 \Theta_3}{\partial Z^2} = \frac{\partial \Theta_3}{\partial \tau_3} \qquad (9)$$

Its solution is given by

$$\Theta_3(Z,\tau_3) = \frac{4}{\pi}\sum_{m=1,3,5...}^{\infty} \frac{1}{m} e^{-(m\pi)^2 \tau_3} \sin(m\pi Z) \qquad (10)$$

The final result, derived in Das *et al.* (1991), in the dimensionless form looks like

$$\Theta_{fc}(R,Z,\tau) = \left[2\sum_{m=1}^{\infty} e^{-\beta_m^2 \tau} \frac{J_0(\beta_m R)}{\beta_m J_1(\beta_m)}\right] \times$$

$$\left[\frac{4}{\pi}\sum_{m=1,3,5...}^{\infty} \frac{1}{m} e^{-(m\pi)^2 \tau (b/L)^2} \sin(m\pi Z)\right] \qquad (11)$$

Here Θ_{fc} is the dimensionless temperature in the finite cylinder, L is its length, and $Z=z/L$ is the dimensionless axial location in the ice core. A computer program combining the infinite cylinder program and the slab program is incorporated in Appendix 3 of Das *et al.* (1991).

Computational Scheme

Infinite Cylinder

The program to compute temperature distribution in an infinite cylinder contains two subroutines adopted from Press *et al.* (1986). They compute Bessel functions of the first kind of orders zero and one which are present in equation (4).

For extremely small values of τ (e.g., 0.001, 0.005) we recommend using a large number of terms in the summation to eliminate oscillations in the final values of temperatures due to the sudden application of boundary temperature. For example, for $\tau=0.001$ we used fifty terms and found that oscillations were eliminated with results stable to the fourth place after decimal. This is more than the accuracy needed for practical engineering calculations. For higher values of τ, ten terms are adequate. Increasing or decreasing the number of terms can be easily accomplished in the computer program.

Semi-infinite Cylinder

This program embodies the first program for the infinite cylinder and then adds to it the semi-infinite solid solution. Therefore, it contains an additional subroutine adopted from Press *et al.* (1986) to calculate the error function. The product of the error function and the infinite cylinder solution gives the temperatures for the semi-infinite cylinder.

Finite Cylinder

The computer program for this case contains the first program and the addition of a subsection for calculating the solution of a slab as derived in equation (10). Temperatures for different core lengths can be easily evaluated by simply changing this number in the program. For the slab solution in equation (11) our program is set to m equal to fifty, guaranteeing extreme accuracy in the series summation.

Results and Discussion

Computation of dimensionless temperature profiles for infinite cylinders from equation (4) via the first program are displayed in Figure 2. The curves in this plot represent dimensionless times varying from $\tau=0.001$ to $\tau=2.5$, which cover practically all ranges of time periods and radii of ice cores. Figure 2 shows that by the time τ reaches one, the entire interior of the ice core has warmed to the surface temperature.

Example 1: Consider an ice core 10 cm (4 inches) in diameter. The initial temperature T_0 is -40°C and the surface temperature T_s due to heating during drilling operation is 0°C. We wish to find the temperature at $r=2$ cm (0.8 inches) after 190 seconds.

Thermal diffusivity α for ice is 1.33×10^{-6} m²/s near -40°C (Hutter, 1983). Dimensionless time $\tau = \alpha t/b^2 = 0.101$. Nondimensional radius $R = r/b = 0.4$ from Figure 2. Corresponding to this τ and R, we read nondimensional temperature

$$\text{THIN} = \Theta_1 = \frac{T_s - T}{T_s - T_o} = 0.7 \qquad (12)$$

With $T_s = 0°C$ and $T_o = -40°C$, we obtain $T(r = 2\ cm, t = 190\ sec) = -28°C$.

Nondimensional temperature profiles for a semi-infinite cylinder computed from equation (8) using the second program are shown in Figure 3. This plot was generated for a dimensionless Z coordinate of 0.5, which represents points at a vertical distance of half the radius from the base of the cylinder. For any other vertical position, simply change the Z value in the error function routine of the program to generate plots similar to Figure 2.

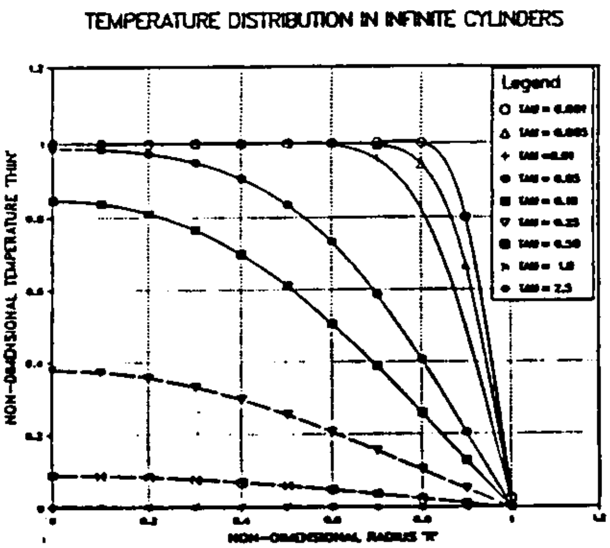

Figure 2. Dimensionless temperature profiles in an infinite cylindrical ice core.

Example 2: Consider an ice core of 15 cm (6 inches) diameter. The initial and surface temperatures are the same as the previous example. Find the temperature at a radial distance $r = 4.5$ cm, and vertical distance $z = 3.75$ cm from the end of the cylinder after 1080 seconds. With these values τ becomes 0.255, $R = 4.5/7.5 = 0.6$ and $Z = 3.75/7.5 = 0.5$. From Figure 3 we read dimensionless temperature

$$\text{THSC} = \Theta_{SC} = \frac{T_s - T}{T_s - T_o} = 0.1 \qquad (13)$$

Solving for the required temperature we obtain $T(r = 4.5\ cm, z = 3.75\ cm, t = 1080\ sec) = -4°C$.

Figure 4 presents the dimensionless temperature profiles in a finite cylinder based on equation (11), which has been incorporated into the third program. The curves in Figure 4 are computed for a nondimensional Z coordinate of 0.2. We have selected a short cylinder with a length to radius (L/b) aspect ratio of two to demonstrate the end effects. For other aspect ratios and Z values, it is simple to change these parameters in the program.

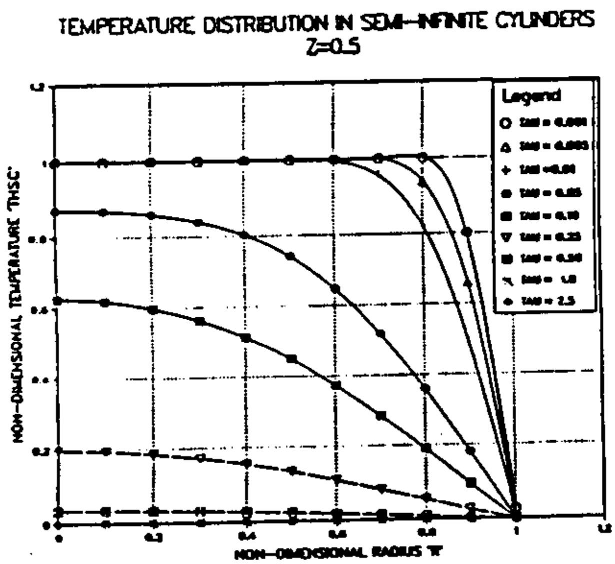

Figure 3. Dimensionless temperature profiles in a semi-infinite cylindrical ice core.

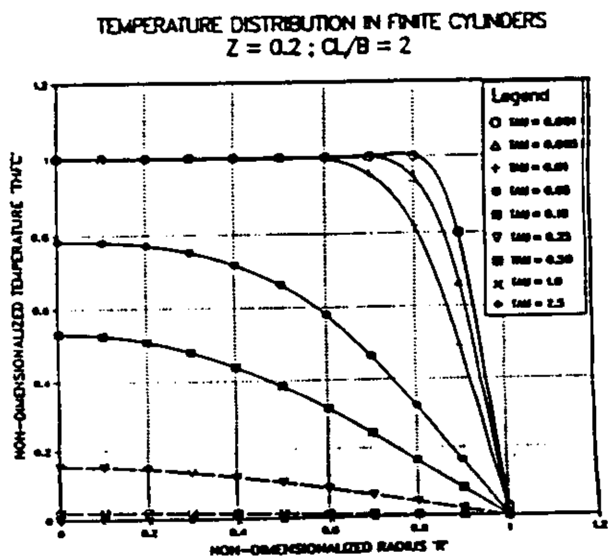

Figure 4. Dimensionless temperature profiles in a finite cylindrical ice core.

Example 3: Consider an ice core of 20 cm (8 inches) diameter and of the same length. Find the temperature at r = 8 cm and z = 4 cm after 375 seconds if the initial and surface temperatures are the same as in Example 1. From given data τ is equal to 0.05. $R=8/10=0.8$ and $Z=4/20=0.2$. From Figure 4 we read

$$THFC=\Theta_{FC}=\frac{T_s-T}{T_s-T_o}=0.325 \tag{14}$$

which gives

$T(r = 8\,cm, z = 4\,cm, t = 375\,sec) = -13°C$.

Limitations

Analytical results obtained in this paper are based on assumptions of homogeneous and linear heat conduction. In order to apply the product solution technique under this condition, one must assume that all surfaces are at the same constant temperature T_s. However, this is not a good assumption for actual cases in the ice field where temperatures vary radially and along the depth. Therefore, a more sophisticated method is necessary to accurately model the actual field conditions.

Recommendations

Presently, one of our graduate students is working on finite element modeling to predict the temperature field. This approach is extremely versatile to handle different types of boundary conditions, namely, convection, heat flux, and variable temperature, which present analytical models cannot simulate. When completed, it would also be able to predict temperatures in the drill casing and the ice surrounding the core, which will enable us to determine heat loss from the drilling fluid into the surrounding ice field.

Conclusions

From Figure 2 we conclude that an ice core under infinite cylinder assumption becomes uniformly warmed to its surface temperature by the time dimensionless time τ attains a value slightly higher than 0.50.

For the second case, shown in Figure 3, heat transfer takes place for a semi-infinite cylindrical core radially as well as axially from the base. Due to the extra conduction from the base in the axial direction, which was not present in the case of an infinite cylinder, we observe faster warming of the core.

From Figure 4 we observe that the interior of the core warms to its surface temperature by the time dimensionless time τ reaches a value of 0.5. Superposition of temperature profiles for $\tau=0.05$ and 0.10 from these three figures clearly shows that the finite cylinder warms faster than the previous two cases due to axial conduction at both ends. Note that the axial location and the aspect ratio are influential parameters. The trend is correct and this last case is more appropriate for ice cores.

At present, we are using the analytical solutions presented herein to validate the finite element program, which is under development. Furthermore, we can use these analytical approaches, which are much simpler than the finite element approach, to obtain quick approximate results when they are necessary at preliminary stages. These analytical models are much simpler to run and less expensive as far as computing resources are concerned compared to a finite element program.

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