ANALYTICAL SOLUTIONS FOR TEMPERATURE DISTRIBUTION IN ICE CORES

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PICO
Polar Ice Coring Office

Polar Ice Coring Office
University of Alaska Fairbanks
Fairbanks, Alaska 99775-1710

PICO
TR 91-3

March 1991

PICO is operated by the University of Alaska Fairbanks under contract to the Naitonal Science Foundation, Division of Polar Programs.

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1. INTRODUCTION

The Polar Ice Coring Office (PICO), operated by the University of Alaska Fairbanks for the National Science Foundation, is charged with the development and operation of ice coring drills and augers for scientific research. In this capacity PICO has developed expertise in lightweight drills and augers and continues the research on applying state-of-the-art materials and techniques for ice coring devices. Many nations now have designated programs to sample deep within the world's ice caps, primarily to study evidence of past climate variations and the properties and movements of glaciers. PICO has developed a deep ice coring drill and has started a three-year project in the summer of 1990 to core through the Greenland ice cap. PICO has also been tasked to plan a similar Antarctic coring program for west Antarctica to begin in the 1992 austral summer. Many foreign nations are also conducting similar programs.

Common to all of the deep ice sampling devices to date has been the use of thousands of gallons of drilling fluids in a designated pristine environment, such as diesel fuels, trichloroethylene, fluorocarbons, etc. PICO has conducted a number of studies and published the results (Gosink, 1989; Gosink *et al.*, 1989) to search for a more environmentally acceptable fluid for use in the Greenland project. A search for an alternative deep ice coring drill which would be both environmentally safe and effective is continuing at PICO.

Proenza *et al.* (1990) have introduced the conceptual design for a hot water-mechanical drill while describing shallow and deep ice coring devices developed by PICO. The main objective is to develop such a drill that can core through thick ice caps in far less time than it now takes, without the use of drilling fluids. We are also aware that new protocols are under discussion that may prohibit the use of conventional drilling fluids or the requirement to pump them out of the wells, which may run into thousands of gallons, and remove them from the particular polar region.

The internal flow geometry of the hot water coring drill is shown schematically in Figure 1. It involves hot water flow in the outer annulus that melts the ice near the tip of cutting tools and assists in drilling at a faster rate. The cooler water flowing through the inner annulus carries the chips up and eventually melts them. The cooler and warmer water are circulated via hoses to the top of the hole where the cooler water goes through a heater and is injected back through the hot water hose.

It should be recognized that this is a new technology. As Rinaldi *et al.* (1990) report, although hot water drilling has been successful, a coring device of this type has not been developed. A very important factor in this design is to use an extremely low thermal conductivity material for the innermost cylinder surrounding the core. Other alternatives are the use of double-wall pipe with air gap or high quality insulation to isolate the core and to minimize the heat penetration. Before we can design a prototype hot water-mechanical drill, we must have an opportunity to conduct an analytical study of the fluid mechanics and heat transfer considerations for this concept. The present study constitutes the basis for analyzing the heat transmission into the ice core, before the design and field test of such a drilling

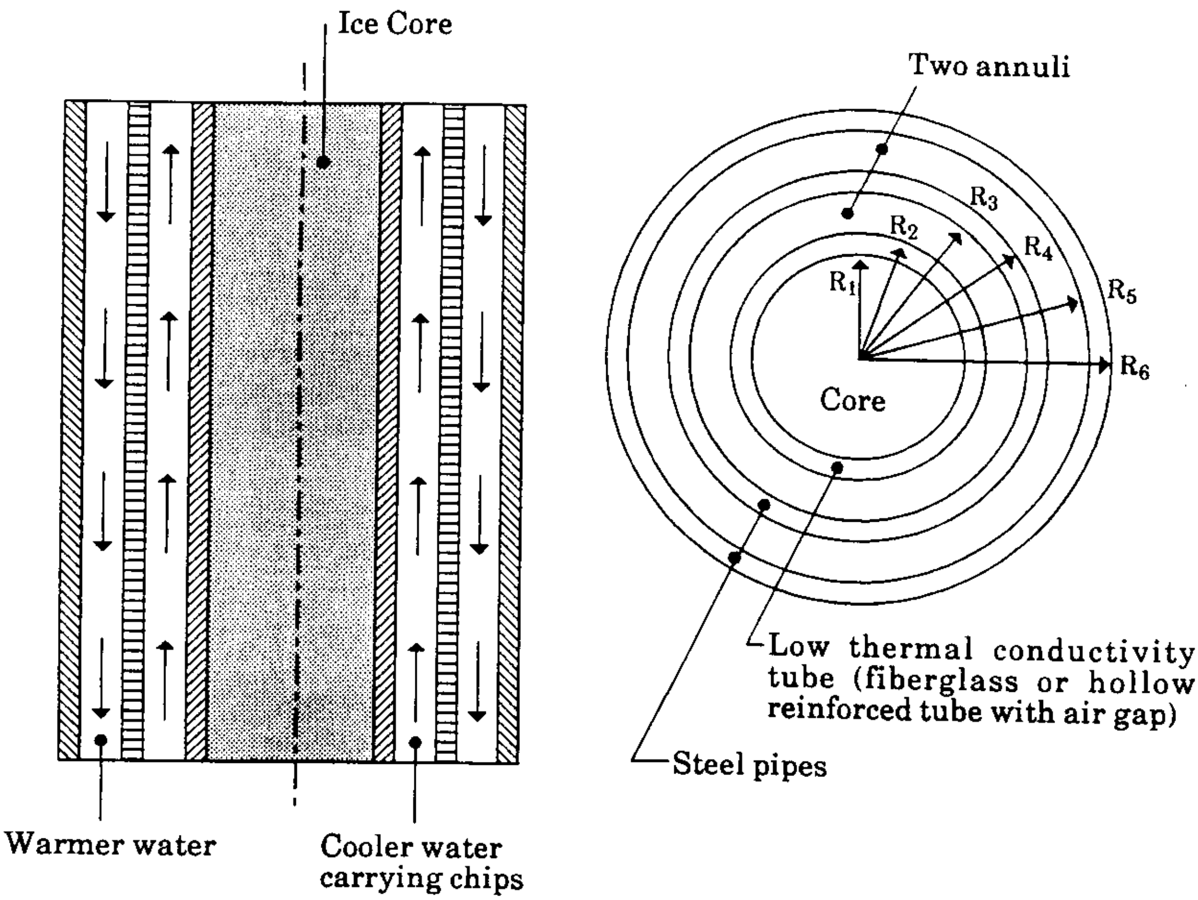

Figure 1. Internal flow geometry of the thermo-mechanical coring drill.

device. We estimate the time to design, test, refine and finalize such a device to be two to five years.

During the first phase of the study, we have focused on one fundamental problem. That is to determine the interior temperature distribution in a cylindrical ice core subjected to different temperatures at the boundary due to the circulation of hot water. Treating the core in three ways: (1) an infinite cylinder; (2) a semi-infinite cylinder; and (3) a finite cylinder, we have developed theories to predict how temperature distributions are changing with time. We wish to find how fast the core is warming up, and what is the risk of melting the core. The three methods are described in the next section. Using them one can determine the minimum core diameter that is obtainable through thermal coring techniques without jeopardizing the interior region of the core due to excessive penetration of heat.

2. ANALYTICAL METHODS

2.1. Infinite Cylinder Approach

Assuming constant properties, no heat generation and no dependence on z and ø, the appropriate form of the heat conduction equation can be written from Ozisik (1980).

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

The boundary and initial conditions are

$T(b,t) = T_s$, the surface temperature (2a)
$T(r,o) = T_o$, the initial temperature (2b)

Figure 2 shows the geometry of the infinite cylinder. Let us nondimensionalize the governing equation (1) and boundary and initial conditions (2a) and (2b) using the following dimensionless variables.

$$\Theta_1 = \frac{T_s - T}{T_s - T_o}; \; R = \frac{r}{b}; \; \tau = \frac{\alpha t}{b^2} = \text{dimensionless time or Fourier number} = F_o \tag{3}$$

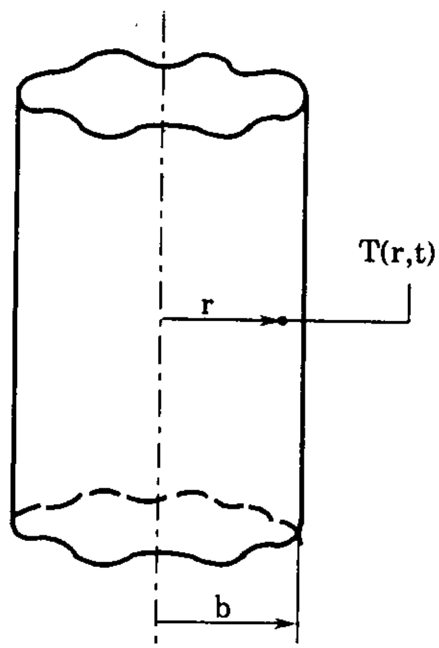

Figure 2. Infinite cylinder geometry.

Substituting the variables in equation (3) into (1) and simplifying, we obtain the nondimensional form of the governing equation.

$$\frac{\partial^2\Theta_1}{\partial r^2} + \frac{1}{r}\frac{\partial^2\Theta_1}{\partial r} = \frac{\partial\Theta_1}{\partial\tau} \tag{4}$$

Similarly, by substituting the variables defined via equation (3) into equation (2a) and (2b), we get the nondimensionalized boundary and initial conditions.

Θ_1 at surface where $R_b = 1$ is zero, because $T = T_s$ (5a)

Initial Θ_1 at time $t = o$ ($\tau = 0$) is $\Theta_{1i} = 1$, because $T = T_o$ (5b)

The solution of equation (4) with initial and boundary conditions prescribed by equations (5a) and (5b) is presented as equation (3-68) in Ozisik (1980).

$$\Theta_1(R,\tau) = \frac{2\Theta_{1i}}{R_b} \sum_{m=1}^{\infty} e^{-\beta_m^2\tau} \frac{J_o(\beta_m R)}{\beta_m J_1(\beta_m R_b)} \tag{6}$$

Recognize from the dimensionless conduction equation (4) that α has been embedded in τ after nondimensionalization. Boundary and initial conditions from equations (5a) and (5b) show that Θ_{1i} and R_b are equal to unity. Therefore, the final result becomes

$$\Theta_1(R,\tau) = 2 \sum_{m=1}^{\infty} e^{-\beta_m^2\tau} \frac{J_o(\beta_m R)}{\beta_m J_1(\beta_m)} \tag{7}$$

where β_m's are positive roots of the Bessel function $J_o(\beta_m) = 0$. The first ten roots of the Bessel function are listed in White (1972). For roots greater than ten, an equation is given in White. We have included these roots and the equation in the computer program which is listed in Appendix 1 of this report.

2.2. Semi-infinite Cylinder Approach

This is a two-dimensional problem with the governing equation given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8}$$

The solution of this type of problem can be written as the product of two one-dimensional solutions, provided the problem is linear and homogeneous (Myers, 1987). The product solution appears as

$$T(r,z,t) = T_1(r,t)\, T_2(z,t) \tag{9}$$

where $T_1(r,t)$ is the solution of the infinite cylinder approach just illustrated in Section 2.1 and $T_2(z,t)$ is the solution of the heat conduction problem in a semi-infinite solid. Figure 3 displays the geometry of a semi-infinite cylinder. In the dimensionless form, the product solution will be

$$\Theta_{sc}(R,Z,\tau) = \Theta_1(R,\tau)\,\Theta_2(Z,\tau) \tag{10}$$

$\Theta_1(R,\tau)$ is already available from equation (7). $\Theta_2(Z,\tau)$ can be found by the following procedure.

The equation and its corresponding boundary and initial conditions for the semi-infinite solid are

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in } o \le z < \infty \tag{11}$$

$T(o,t) = T_s$, the surface temperature for time $t > o$ (12a)

$T(z,o) = T_o$, the initial temperature in $o \le z < \infty$ (12b)

Let us nondimensionalize the initial temperature in equations (11), (12a) and (12b) using the following variables.

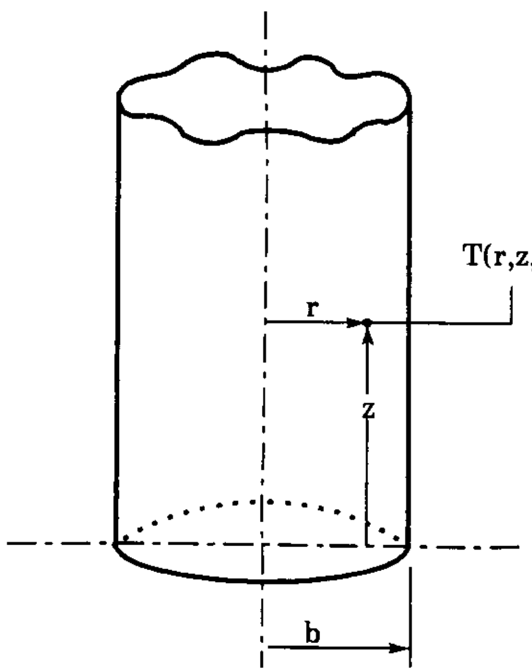

Figure 3. Semi-infinite cylinder geometry.

$$\Theta_2 = \frac{T_s - T}{T_s - T_o}\ ;\ Z = \frac{z}{b};\ \tau = \frac{\alpha t}{b^2} \tag{13}$$

We have used the same length for nondimensionalizing as in equation (3), since semi-infinite body has no fixed length dimension. Substituting terms of equation (13) into (11) and simplifying, we get

$$\frac{\partial^2 \Theta_2}{\partial Z^2} = \frac{\partial \Theta_2}{\partial \tau} \tag{14}$$

Nondimensionalized boundary and initial conditions from equations (12a) and (12b) are

Θ_2 at surface where $Z = o$ is zero, because $T = T_s$ (15a)

Initial Θ_2 at time $t = o$ ($\tau = o$) is $\Theta_{2i} = 1$, because $T = T_o$ (15b)

The solution of equation (14) subject to the conditions given by equations (15a) and (15b) is given in Ozisik (1980) as equation 2-62. In the dimensionless form we can write it as

$$\Theta_2(Z,\tau) = \mathrm{erf}\left(\frac{Z}{\sqrt{4\tau}}\right) \tag{16}$$

Notice that the constant initial temperature T_o that appears as the denominator in Ozisik's solution becomes Θ_{2i} and goes to unity after nondimensionalization and so does α. Now combine equations (16) and (7) to form the final solution of the semi-infinite cylinder problem.

$$\Theta_{SC}(R, Z,\tau) = \mathrm{erf}\left(\frac{Z}{\sqrt{4\tau}}\right) \left[2\sum_{m=1}^{\infty} e^{-\beta_m^2 \tau}\ \frac{J_o(\beta_m R)}{\beta_m J_1(\beta_m)}\right] \tag{17}$$

The computer program to calculate temperatures combining the infinite cylinder program with an error function program is listed in Appendix 2 of this report.

2.3. Finite Cylinder Approach

Figure 4 illustrates the geometry of a finite cylinder. The governing equation for this case is the same as equation (8). the solution for this case can also be obtained as the product of two one-dimensional solutions illustrated earlier in Section 2.2. The problem must be linear and homogeneous. The product solution is

$$T(r,z,t) = T_1(r,t)\ T_3(z,t) \tag{18}$$

where $T_1(r,t)$ is the solution of the infinite cylinder approach already completed in Section 2.1 and $T_3(z,t)$ is the solution of the heat conduction through a slab whose thickness is equal to the height of the cylinder. In the dimensionless form the solution looks like

$$\Theta_{FC}(R,Z,\tau) = \Theta_1(R,\tau)\,\Theta_3(Z,\tau_3) \tag{19}$$

$\Theta_1(R,\tau)$ is available from equation (7). Let us obtain the solution $\Theta_3(Z,\tau_3)$. The mathematical formulation for a slab problem can be written as

$$\frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in } 0 \le z \le L,\ t > 0 \tag{20}$$

Bounday conditions: $T = T_s$ at $z = 0$ and $z = L$ for $t > 0$ (21a)

Initial condition: $T = T_0$ for $t = 0$ in $0 \le z \le L$ (21b)

Nondimensionalize equations (20), (21a) and (21b) using the following variables.

$$\Theta_3 = \frac{T_s - T}{T_s - T_0};\quad Z = \frac{z}{L};\quad \tau_3 = \frac{\alpha t}{L^2} \tag{22}$$

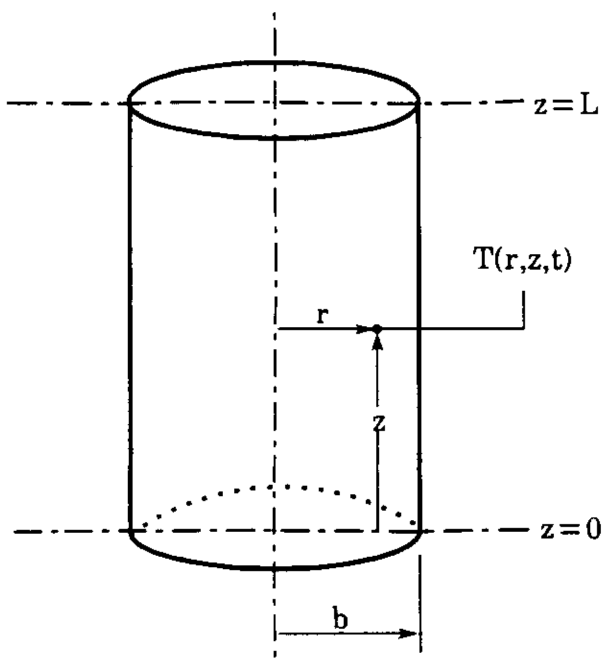

Figure 4. Finite cylinder geometry.

Substituting these nondimensional variables into equation (20) we get

$$\frac{\partial^2\Theta_3}{\partial Z^2}=\frac{\partial\Theta_3}{\partial\tau_3} \qquad (23)$$

Boundary conditions and initial condition from equations (21a) and (21b), when nondimensionalized, yield

$$\Theta_3=0 \text{ at surfaces } Z=0 \text{ and } Z=1\text{, because } T=T_s \qquad (24a)$$

$$\text{Initial } \Theta_3 \text{ at time } t=0(\tau_3=0) \text{ is } \Theta_{3i}=1\text{, because } T=T_0 \qquad \text{in } o\leq Z\leq 1 \qquad (24b)$$

The solution to equation (23) along with conditions (24a) and (24b) is obtained from Ozisik (1980) [Table 2-2, Case #9, and equation (2-36)]. Notice that for this nondimensional case $\propto$, L and the initial temperature Θ_{3i} are all equal to one.

$$\Theta_3(Z,\tau_3)=\sum_{m=1}^{\infty} e^{-\beta_m^2\tau_3}\ (2)\sin\beta_m Z \int_0^1 \sin\beta_m Z'\,dZ' \qquad (25)$$

where β_m's are the roots of $\sin\beta_m=0$ or given as $\beta_m=m\pi$, $m=1,2,3,\ldots$.

Completing the integration in equation (25), the final result from Ozisik (1985) is:

$$\Theta_3(Z,\tau_3)=\frac{4}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m}e^{-(m\pi)^2\tau_3}\sin(m\pi Z) \qquad (26)$$

Notice that the dimensionless time τ_3 can be expressed in terms of τ by the relation $\tau_3=\tau(b/L)^2$. With this substitution the final result, as a product of two solutions, becomes

$$\Theta_{FC}(R,Z,\tau)=\left[2\sum_{m=1}^{\infty}e^{-\beta_m^2\tau}\frac{J_o(\beta_m R)}{\beta_m J_1(\beta_m)}\right]\left[\frac{4}{\pi}\sum_{m=1,3,5,\ldots}^{\infty}\frac{1}{m}e^{-(m\pi)^2\tau(b/L)^2}\sin(m\pi Z)\right] \qquad (27)$$

A computer program combining the infinite cylinder program and the slab program is incorporated in Appendix 3.

3. COMPUTATIONAL SCHEME

3.1 Infinite Cylinder

The first program in the appendix computes temperature distribution in an infinite cylinder. Ten roots of Bessel function from White (1974) have been substituted into the program. Roots greater than ten in the program are evaluated by an equation also from White. Two subroutines adopted from Press *et al.* (1986) compute Bessel functions of the first kind of order zero and one which are present in equation (7).

The results given in Appendix 1 as an example, are for a dimensionless time $\tau=0.05$. Other time periods were calculated simply changing the value of τ in the program. For extremely small values of τ (e.g., 0.001, 0.005) we recommend using a large number of terms, say about a hundred, in the summation to eliminate oscillations in the final values of temperatures. For example, for $\tau=0.001$ we used fifty terms, and found that oscillations were eliminated up to the fourth place after decimal. This is more than the accuracy needed for practical engineering calculations. For higher values of τ, ten terms are adequate. Increasing or decreasing the number of terms can be easily accomplished by simply changing the number in the do-loop of the program.

3.2 Semi-infinite Cylinder

This is presented as the second program in the appendix. This program embodies the first program for the infinite cylinder, and then adds on the semi-infinite solid solution to it. Therefore, we have included an additional subroutine to calculate the error function adopted from Press *et al.* (1986). The product of the error function and the infinite cylinder solution gives the temperatures for semi-infinite cylinder. The sample run given in the appendix is for a nondimensional height $Z=0.5$, meaning the point is at a height equal to half the radius of the cylinder. Temperatures at various heights can be computed by simply changing the input values of Z in the program. A sample output shown in Appendix 2 is for $\tau=0.25$.

3.3 Finite Cylinder

The computer program for this case is presented as Program-3 in Appendix 3. It contains Program-1 as before, and then adds on a subsection for calculating the solution of a slab as derived in equation (26). The sample run given in the appendix is for a nondimensional height $Z=0.5$, which represents a plane at mid-height of the finite cylinder. Temperatures at other heights can be easily evaluated by simply changing this number in the program. For the slab solution we have taken fifty terms in the series summation. The sample output from this program in Appendix 3 is for $\tau=0.1$.

4. RESULTS & DISCUSSION

Computation of dimensionless temperature profiles for infinite cylinders from equation (7) via Program-1 of the appendix are displayed in Figure 5. The curves in this plot represent dimensionless times varying from $\tau=0.001$ to $\tau=2.5$, which cover practically all ranges of time periods and radii of ice cores. Figure 5 shows that by the time τ reaches one, the entire interior of the ice core has been heated up the surface temperature. Beyond that no further changes in the temperature occur. The top two curves representing very small time $\tau=0.001$ and $\tau=0.005$ must be computed using a large number of terms in the series in equation (7), otherwise the final values may exhibit an oscillatory behavior.

Example 1: Consider an ice core 10 cm (4 inches) in diameter. The initial temperature T is -40°C and the surface temperature T_s due to heating during drilling operation is 0°C. We wish to find temperature at r = 2 cm (5 inches) after 190 seconds.

Thermal diffusivity α for ice is 1.33×10^{-6} m²/s at -55°C (Hutter, 1983). Dimensionless time $\tau=\alpha t/b^2=0.101$. Nondimensional radius $R=r/b=0.4$ from Figure 5. Corresponding to this τ and R we read nondimensional temperature

$$\mathrm{THIN}=\frac{T_s-T}{T_s-T_o}=0.7;\ \text{with } T_s=0°\mathrm{C} \text{ and } T_o=-40°\mathrm{C}$$

We obtain $T(r=2\text{ cm}, t=190\text{ sec})=-28°\mathrm{C}$.

Nondimensional temperature profiles for a semi-infinite cylinder computed from equation (17) using Program-2 of the appendix are shown in Figure 6. This plot was generated for a dimensionless Z coordinate of 0.5 which represents points at a vertical distance of half the radius from the base of the cylinder. For any other vertical position simply change the Z value in the error function section of program-2, and generate plots similar to Figure 6.

Example 2: Consider an ice core of 15 cm (6 inches) diameter. The initial and surface temperatures are the same as the previous example. Find the temperature at a radial distance r = 4.5 cm, and vertical distance z = 7.5 cm from the end of the cylinder after 1080 seconds. With these values τ becomes 0.255, $R=4.5/7.5=0.6$ and $Z=7.5/15=0.5$. From Figure 6 we read dimensionless temperature

$$\mathrm{THSC}=\Theta=\frac{T_s-T}{T_s-T_o}=0.2$$

Solving for the required temperature we obtain

$$T(r=4.5\text{ cm}, z=7.5\text{ cm}, t=1080\text{ sec})=-8°\mathrm{C}.$$

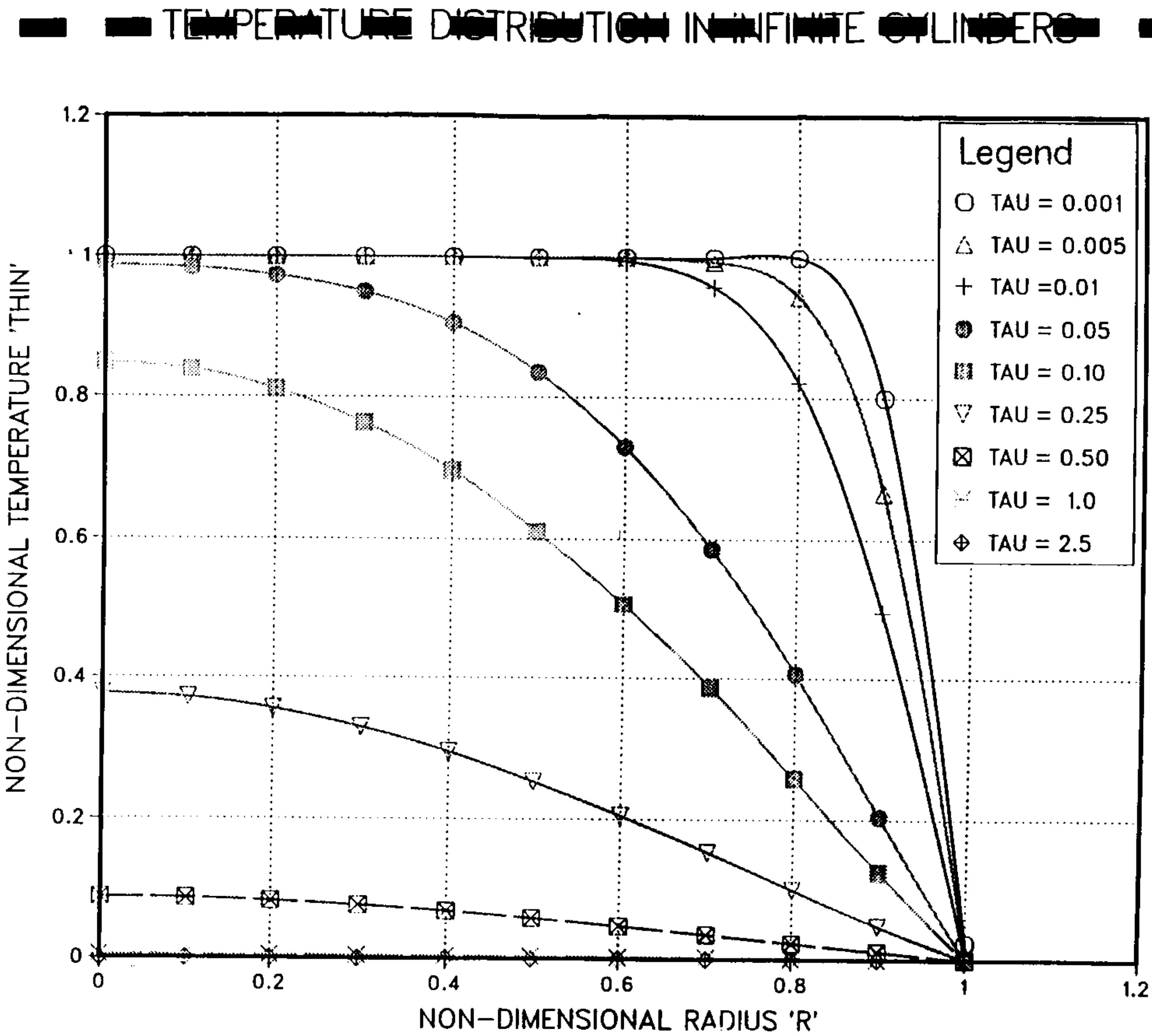

Figure 5. Dimensionless temperature profiles in an infinite cylindrical ice core.

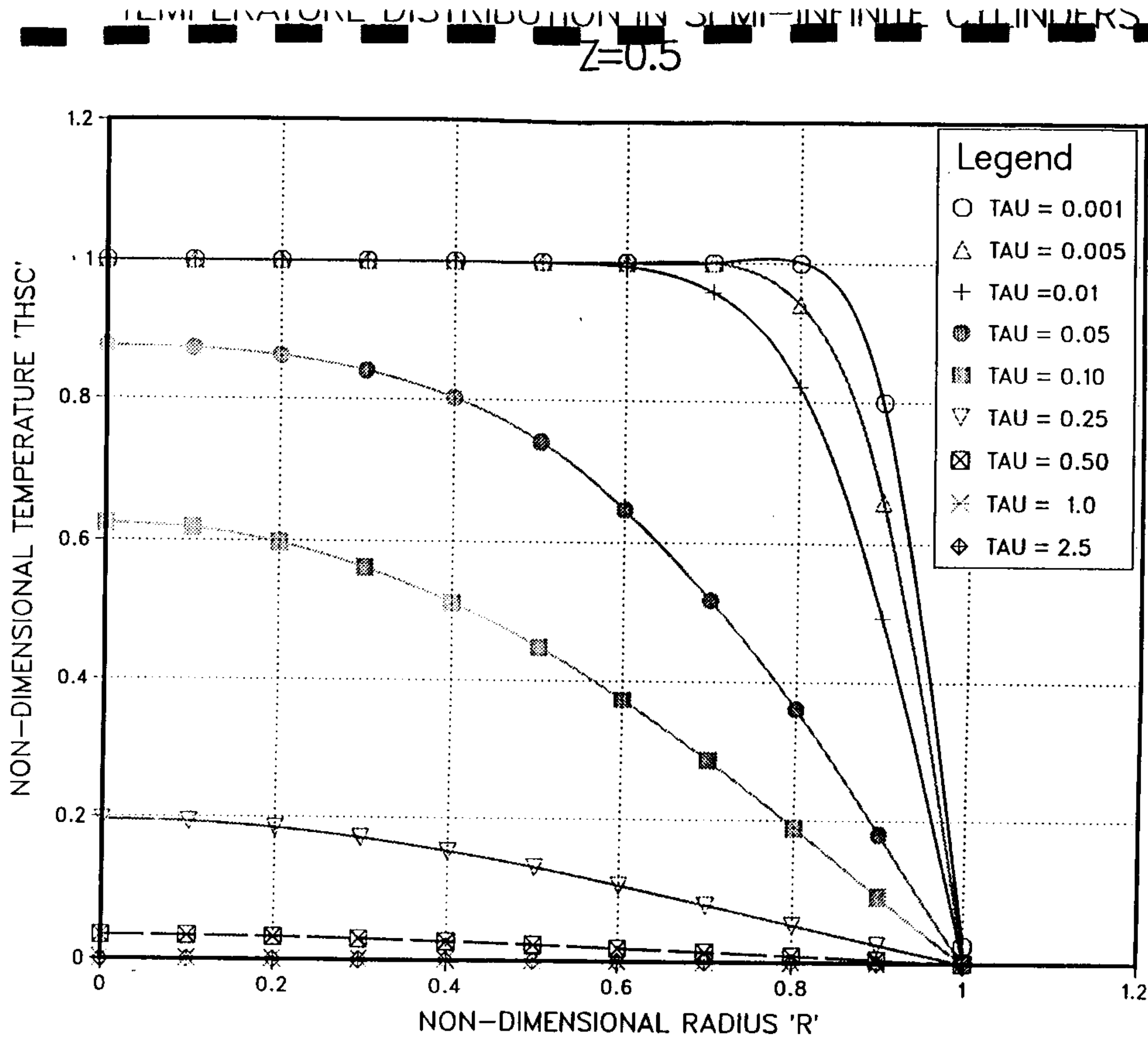

Figure 6. Dimensionless temperature profiles in a semi-infinite cylindrical ice core.

Figure 7 presents the dimensionless temperature profiles in a finite cylinder based on equation (27) which has been incorporated in Program-3 of the appendix. The curves in Figure 7 are computed for a nondimensional Z coordinate of 0.5 which represents points located exactly at half the height of the cylinder. For points on a plane at any other vertical distance, simply change the Z value in the slab calculation section of Program-3.

Example 3: Consider an ice core of 20 cm (8 inches) diameter and 3 m (9.84 ft) length. Find the temperature at $r = 8$ cm and $z = 1.5$ m after 375 seconds if the initial and surface temperatures are the same as in Example 1. From given data τ is equal to 0.05. $R=8/10=0.8$ and $Z=1.5/3.0=0.5$. From Figure 7 we read

$$\mathrm{THFC} = \frac{T_S - T}{T_S - T_o} = 0.40$$

which gives $T(r=8\text{ cm}, z=1.5\text{ m}, t=375\text{ sec}) = -16°\text{C}$.

5. CONCLUSIONS AND RECOMMENDATIONS

The following conclusions can be drawn from this study.

From Figure 5 we conclude that an ice core under infinite cylinder assumption becomes uniformly warmed to the surface temperature by the time the dimensionless time τ attains a value slightly higher than 0.05. Beyond this time period the excess temperature between the surface and the interior goes to zero and no further heat conduction takes place inside the core.

For the second case the heat transfer takes place for a semi-infinite cylindrical core radially as well as axially from the base. The base due to the extra conduction in the axial direction which was not present in case of an infinite cylinder, we observe faster warming of the core in Figure 6 in comparison with Figure 5. From Figure 6 we note that the entire core gets warmed up to the surface temperature by the time the dimensionless time τ becomes about 0.5.

From Figure 7 we observe that the interior of the core gets warmed to the surface temperature by the time the dimensionless time τ reaches a value slightly higher that 0.25. Comparisons of temperature profiles at various times in the figure with Figures 5 and 6 clearly show that the cylinder is warming faster than the previous two cases.

Analytical results obtained in this report are based on homogeneous heat conduction problems. In order to apply the product solution technique under this condition, one must assume that all surfaces are at the same constant temperature T_S. However, this is not a good assumption for actual cases in the ice field where temperatures vary radially and along the depth. Therefore a more sophisticated method is necessary to accurately model the actual field conditions.

We recommend that finite element modelling be undertaken to predict the temperature field. This approach should be made versatile to handle different types

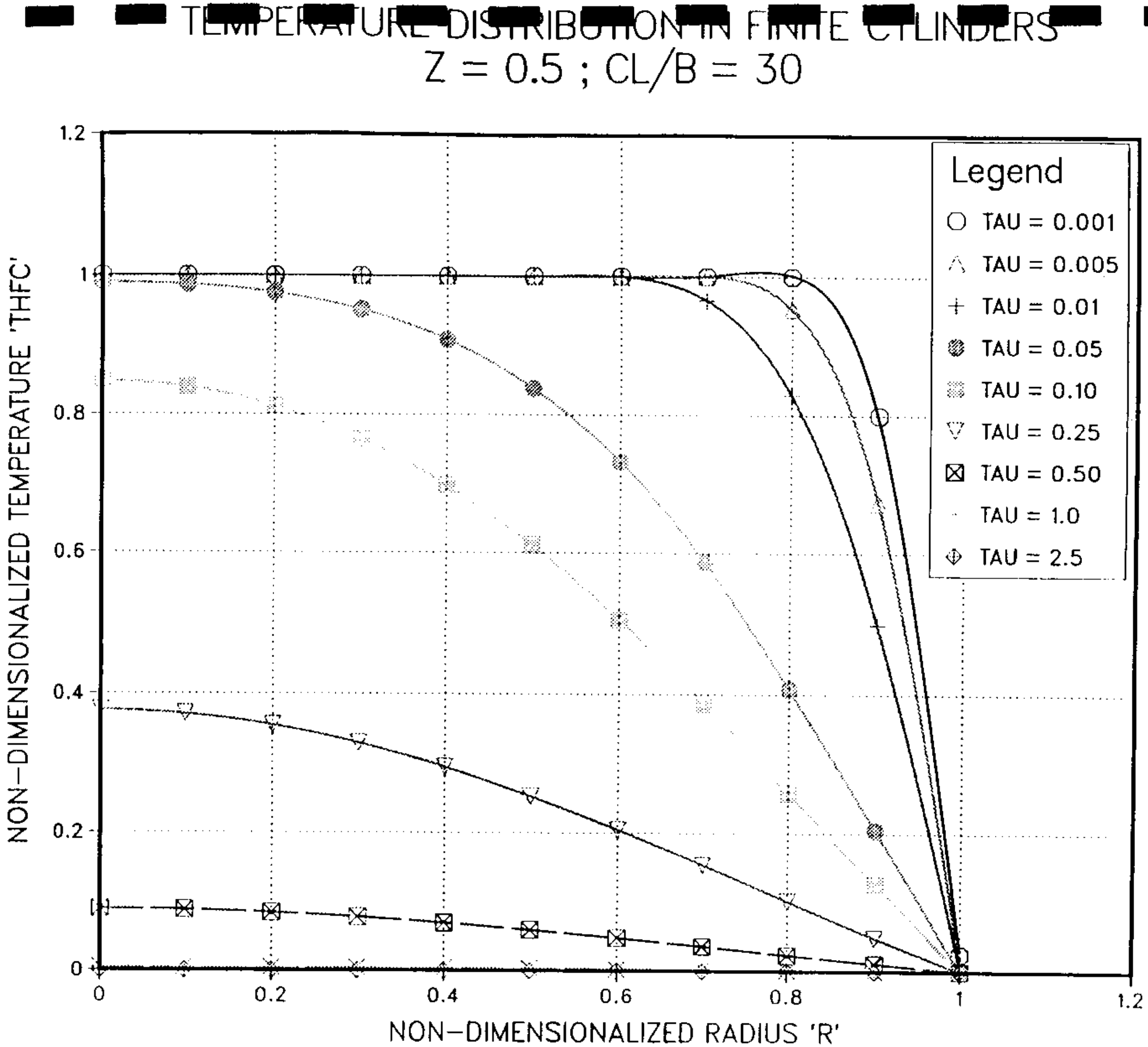

Figure 7. Dimensionless temperature profiles in a finite cylindrical ice core.

of boundary conditions, namely; 1. convection, 2. heat flux, 3. variable temperature which the present analytical models can not simulate. It should also be able to predict temperatures in the drill casing and the ice surrounding the core which will enable us to determine the heat loss from the drilling fluid into the surrounding ice field.

We shall use the analytical solutions developed here to validate the finite element program during its evolution. Furthermore, we can use these analytical approaches which are much simpler than the finite element approach to obtain quick approximate results when they are necessary at preliminary stages. These analytical models are much simpler to run and less expensive as far as computing resources are concerned compared to a finite element program.

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APPENDIX 1

PROGRAM-1 FOR THE INFINITE CYLINDER THEORY AND A SAMPLE OUTPUT

PROGRAM-1

```
C        PROGRAM TO CALCULATE TEMPERATURES IN AN INFINITE CYLINDER
         DIMENSION BM(50),THIN(50)
         OPEN(UNIT=10, FILE='THIN.OUT', STATUS='NEW')
C        EXPLANATION OF INPUT PARAMETERS
C        B = RADIUS AT THE SURFACE
C        BO = NON-DIMENSIONALIZED RADIUS AT SURFACE
C        BM = BETA WITH SUBSCRIPT M
C        R = VARYING RADIUS
C        TAU = NON-DIMENSIONALIZED TIME = ALPHA*TIME/B**2
C        THIN = NON-DIMENSIONALIZED TEMP FOR INFINITE CYLINDER
         BO = 1.0
C        HERE ARE THE VALUES FOR BETA
         BM(1) =2.4048
         BM(2) =5.5201
         BM(3) = 8.6537
         BM(4) = 11.7915
         BM(5) = 14.9309
         BM(6) =18.0711
         BM(7) = 21.2116
         BM(8) = 24.3525
         BM(9) = 27.4935
         BM(10) = 30.6346
         DO 10 J = 11,50
         BM(J) = REAL(4*J-1)*3.1416/4.0
   10    CONTINUE
         TAU = 0.05
         R = 0.0
1000     SUM=0.0
         DO 200 K = 1,50
         W = BM(K)*R
         Z = BM(K)*BO
         C = (EXP(-BM(K)*BM(K)*TAU))*(BESSJ0(W))/(Z*BESSJ1(Z))
C        WRITE(10,*)C,BM(K),BESSJ0(W),BESSJ1(Z)
         SUM = SUM + C
C        ALL PRINT STATEMENTS FOR VIEWING INTERMEDIATE
C        RESULTS ON THE SCREEN
C        PRINT*,SUM
         THIN(K) = 2.0*SUM
C        PRINT *,'THIN =',R,THIN(K)
  200    CONTINUE
         WRITE(10,*)R,THIN(50)
         R = R + 0.1
         IF (R.LE.1.1) GO TO 1000
         STOP
         END
         FUNCTION BESSJ0(X)
C        RETURNS THE BESSEL FUNCTION J0(X) FOR ANY REAL X
         REAL*8 Y,P1,P2,P3,P4,P5,
     *        Q1,Q2,Q3,Q4,Q5,
     *        R1,R2,R3,R4,R5,R6,
     *        S1,S2,S3,S4,S5,S6
         DATA P1,P2,P3,P4,P5/
     *           1.D0,-.1098628627D-2,
```

```
*        .2734510407D-4,
*        -.2073370639D-5,.2093887211D-6/,
*        Q1,Q2,Q3,Q4,Q5/-.1562499995D-1,
*        .1430488765D-3,-.6911147651D-5,
*        .7621095161D-6,-.934945152D-7/
      DATA R1,R2,R3,R4,R5,R6
*         /57568490574.D0,
*        -13362590354.D0,651619640.7D0,
*          -11214424.18D0,77392.33017D0,-184.9052456D0/,
*        S1,S2,S3,S4,S5,S6/57568490411.D0,1029532985.D0,
*        9494680.718D0,59272.64853D0,267.8532712D0,1.D0/
      IF(ABS(X).LT.8.)THEN
      Y=X**2
      BESSJ0=(R1+Y*(R2+Y*(R3+Y*(R4+Y*(R5+Y*R6)))))
*         /(S1+Y*(S2+Y*(S3+Y*(S4+Y*(S5+Y*S6)))))
      ELSE
      AX=ABS(X)
      Z=8./AX
      Y=Z**2
      XX=AX-.785398164
      BESSJ0=SQRT(.636619772/AX)
*     *(COS(XX)*(P1+Y*P2+Y*(P3+Y*(P4+Y
*     *P5))))-Z*SIN(XX)*(Q1+Y*(Q2+Y*(Q3+Y*(Q4+Y*Q5))))
      ENDIF
      RETURN
      END

      FUNCTION BESSJ1(X)
C      RETURNS THE BESSEL FUNCTION J1(X) FOR ANY REAL X
      REAL*8 Y,P1,P2,P3,P4,P5,
*     Q1,Q2,Q3,Q4,Q5,
*          R1,R2,R3,R4,R5,R6,
*     S1,S2,S3,S4,S5,S6
      DATA R1,R2,R3,R4,R5,R6/72362614232.D0,
*        -7895059235.D0,242396853.1D0,
*     -2972611.439D0,15704.48260D0,
*     -30.16036606D0/,

*     S1,S2,S3,S4,S5,S6/144725228442.D0,
*     2300535178.D0,
*     18583304.74D0,99447.43394D0,
*     376.9991397D0,1.D0/
      DATA P1,P2,P3,P4,P5/1.D0,.183105D-2,
*     -.3516396496D-4,.2457520174D-5,
*     -.240337019D-6/,Q1,Q2,Q3,Q4,Q5/
*     .04687499995D0,-.2002690873D-3,
*     .8449199096D-5,-.88228987D-6,.105787412D-6/
      IF(ABS(X).LT.8.)THEN
      Y=X**2
      BESSJ1=X*(R1+Y*(R2+Y*
*     (R3+Y*(R4+Y*(R5+Y*R6)))))
*     /(S1+Y*(S2+Y*(S3+Y*(S4+Y*(S5+Y*S6)))))
      ELSE
```

```
      AX=ABS(X)
      Z=8./AX
      Y=Z**2
      XX=AX-2.356194491
      BESSJ1=SQRT(.636619772/AX)
     * *(COS(XX)*(P1+Y*(P2+Y*(P3+Y*(P4+Y
     *      *P5))))-Z*SIN(XX)*(Q1+Y*
     * (Q2+Y*(Q3+Y*(Q4+Y*Q5)))))
     * *SIGN(1.,X)
      ENDIF
      RETURN
      END
```

OUTPUT FOR TAU = 0.05

R	THIN
0.0000000E+00	0.9871099
0.1000000	0.9837919
0.2000000	0.9724172
0.3000000	0.9487334
0.4000000	0.9059071
0.5000000	0.8355553
0.6000000	0.7300138
0.7000000	0.5856681
0.8000001	0.4062860
0.9000001	0.2044779
1.000000	2.3027134E-04

APPENDIX 2

PROGRAM-2 FOR THE SEMI-INFINITE CYLINDER THEORY AND A SAMPLE OUTPUT

PROGRAM-2

```
C       PROGRAM TO CALCULATE TEMPERATURES IN AN SEMI-INFINITE CYLINDER
        DIMENSION BM(50),THIN(50),THSC(50)
        OPEN(UNIT=10, FILE='THSC.OUT', STATUS='NEW')
C       EXPLANATION OF INPUT PARAMETERS
C       B = RADIUS AT THE SURFACE
C       BO = NON-DIMENSIONALIZED RADIUS AT THE SURFACE
C       BM = BETA WITH SUBSCRIPT M
C       ZCAP = NON-DIMENSIONALIZED LENGTH = Z/B
C       TAU = NON-DIMENSIONALIZED TIME = ALPHA*TIME/B**2
C       THSC = NON-DIMENSIONALIZED TEMP FOR SEMI-INFINITE CYLINDER
C       THIN = NON-DIMENSIONALIZED TEMP FOR INFINITE CYLINDER
        BO = 1.0
C       HERE ARE THE VALUES OF BETA
        BM(1) = 2.4048
        BM(2) = 5.5201
        BM(3) = 8.6537
        BM(4) = 11.7915
        BM(5) = 14.9309
        BM(6) = 18.0711
        BM(7) = 21.2116
        BM(8) = 24.3525
        BM(9) = 27.4935
        BM(10) =30.6346
        DO 10 J = 11,50
        BM(J) = REAL(4*J-1)*3.1416/4.0
   10   CONTINUE
        TAU = 0.25
        R = 0.0
1000    SUM =0.0
        DO 200 K = 1,50
        W = BM(K)*R
        U = BM(K)*BO
        C = EXP(-BM(K)*BM(K)*TAU)*(BESSJ0(W))/(U*BESSJ1(U))
        SUM = SUM + C
C       ALL PRINT STATEMENTS FOR VIEWING THE INTERMEDIATE
C       RESULTS ON THE SCREEN
C       PRINT*,SUM
        THIN(K) = 2.0*SUM
        ZCAP = 0.5
C       PRINT*,THIN(K)
C       CALCULATION OF ERROR FUNCTION
C       HERE Z REPRESENTS THE ARGUMENT OF THE
C       COMPLIMENTARY ERROR FUNCTION
        P=ZCAP/(SQRT(4.0*TAU))
C       PRINT*,P
        Z=ABS(P)
        T1=1./(1.+0.5*Z)
        ERFCC=T1*EXP(-Z*Z-1.26551223+
     *  T1*(1.00002368+T1*(.37409196+
     *  T1*(.09678418+T1*(-.18628806+
     *  T1*(.27886807+T1*(-1.13520398+
     *  T1*(1.48851587+T1*(-.82215223+T1*.17087277)))))))))
        IF (P.LT.0.) ERFCC=2.-ERFCC
```

```
      ERF= 1-(ERFCC)
C     PRINT *,ERF
C     CALCULATION OF TEMP
      THSC(K)=ERF*THIN(K)
C     PRINT *,'THSC=',R,THIN(K),THSC(K)
200   CONTINUE
      WRITE(10,*)R,THSC(50)
      R=R+0.1
      IF(R.LE.1.1) GO TO 1000
      STOP
      END
      FUNCTION BESSJ0(X)
C     RETURNS THE BESSEL FUNCTION J0(X) FOR ANY REAL X
      REAL*8 Y,P1,P2,P3,P4,P5,
     *     Q1,Q2,Q3,Q4,Q5,
     *     R1,R2,R3,R4,R5,R6,
     *     S1,S2,S3,S4,S5,S6
      DATA P1,P2,P3,P4,P5/
     *       1.D0,-.1098628627D-2,
     *     .2734510407D-4,
     *     -.2073370639D-5,.2093887211D-6/,
     *     Q1,Q2,Q3,Q4,Q5/-.1562499995D-1,
     *     .1430488765D-3,-.6911147651D-5,
     *     .7621095161D-6,-.934945152D-7/
      DATA R1,R2,R3,R4,R5,R6
     *      /57568490574.D0,
     *     -13362590354.D0,651619640.7D0,
     *     -11214424.18D0,77392.33017D0,-184.9052456D0/,
     *     S1,S2,S3,S4,S5,S6/57568490411.D0,1029532985.D0,
     *      9494680.718D0,59272.64853D0,267.8532712D0,1.D0/
      IF(ABS(X).LT.8.)THEN
      Y=X**2
      BESSJ0=(R1+Y*(R2+Y*(R3+Y*(R4+Y*(R5+Y*R6)))))
     *      /(S1+Y*(S2+Y*(S3+Y*(S4+Y*(S5+Y*S6)))))
      ELSE
      AX=ABS(X)
      Z=8./AX
      Y=Z**2
      XX=AX-.785398164
      BESSJ0=SQRT(.636619772/AX)
     * *(COS(XX)*(P1+Y*P2+Y*(P3+Y*(P4+Y
     * *P5))))-Z*SIN(XX)*(Q1+Y*(Q2+Y*(Q3+Y*(Q4+Y*Q5))))
      ENDIF
      RETURN
      END

      FUNCTION BESSJ1(X)
C     RETURNS THE BESSEL FUNCTION J1(X) FOR ANY REAL X
      REAL*8 Y,P1,P2,P3,P4,P5,
     * Q1,Q2,Q3,Q4,Q5,
     *     R1,R2,R3,R4,R5,R6,
     * S1,S2,S3,S4,S5,S6
      DATA R1,R2,R3,R4,R5,R6/72362614232.D0,
```

```
     *        -7895059235.D0,242396853.1D0,
     *  -2972611.439D0,15704.48260D0,
     *  -30.16036606D0/,
     *  S1,S2,S3,S4,S5,S6/144725228442.D0,
     *  2300535178.D0,
     *  18583304.74D0,99447.43394D0,
     *  376.9991397D0,1.D0/
        DATA P1,P2,P3,P4,P5/1.D0,.183105D-2,
     *  -.3516396496D-4,.2457520174D-5,
     *  -.240337019D-6/,Q1,Q2,Q3,Q4,Q5/
     *  .04687499995D0,-.2002690873D-3,
     *  .8449199096D-5,-.88228987D-6,.105787412D-6/
        IF(ABS(X).LT.8.)THEN
        Y=X**2
        BESSJ1=X*(R1+Y*(R2+Y*
     *  (R3+Y*(R4+Y*(R5+Y*R6)))))
     *  /(S1+Y*(S2+Y*(S3+Y*(S4+Y*(S5+Y*S6)))))
        ELSE
        AX=ABS(X)
        Z=8./AX
        Y=Z**2
        XX=AX-2.356194491
        BESSJ1=SQRT(.636619772/AX)
     *  *(COS(XX)*(P1+Y*(P2+Y*(P3+Y*(P4+Y
     *        *P5))))-Z*SIN(XX)*(Q1+Y*
     *  (Q2+Y*(Q3+Y*(Q4+Y*Q5)))))
     *  *SIGN(1.,X)
        ENDIF
        RETURN
        END
```

OUTPUT FOR TAU = 0.25

R	THSC
0.0000000E+00	0.1961487
0.1000000	0.1933395
0.2000000	0.1850296
0.3000000	0.1715672
0.4000000	0.1535167
0.5000000	0.1316355
0.6000000	0.1068408
0.7000000	8.0169059E-02
0.8000001	5.2727610E-02
0.9000001	2.5642278E-02
1.000000	2.5864151E-06

APPENDIX 3

PROGRAM-3 FOR THE FINITE CYLINDER THEORY AND A SAMPLE OUTPUT

PROGRAM-3

```
C         PROGRAM TO CALCULATE TEMPERATURES IN A FINITE CYLINDER
          DIMENSION BM(50),THIN(50),THSL(50)
          OPEN(UNIT=10, FILE='THFC.OUT', STATUS='NEW')
C         EXPLANATION OF INPUT PARAMETERS
C         B = RADIUS AT THE SURFACE
C         BO = NON-DIMENSIONALIZED RADIUS AT THE SURFACE
C         BM = BETA WITH SUBSCRIPT M
C         CL = LENGTH OF THE CYLINDER
C         R =  VARYING RADIUS
C         TAU = NON-DIMENSIONALIZED TIME = ALPHA*TIME/B**2
C         TAU3 = NON-DIMENSIONALIZED TIME = ALPHA*TIME/CL**2
C         THIN = NON-DIMENSIONALIZED TEMP OF THE INFINITE CYLINDER
C         THSL = NON-DIMENSIONALIZED TEMP OF SLAB
C         THFC = NON-DIMENSIONALIZED TEMP OF FINITE CYLINDER
C         Z = NON-DIMENSIONALIZED LENGTH = z/CL
          BO = 1.0
C         HERE ARE THE VALUES FOR BETA
          BM(1) =2.4048
          BM(2) =5.5201
          BM(3) = 8.6537
          BM(4) = 11.7915
          BM(5) = 14.9309
          BM(6) = 18.0711
          BM(7) = 21.2116
          BM(8) = 24.3525
          BM(9) = 27.4935
          BM(10) = 30.6346
          DO 10 J = 11,50
          BM(J) = REAL (4*J-1)*3.1416/4.0
   10     CONTINUE
          TAU = 0.10
C         CALCULATION FOR TEMPERATURE IN A SLAB
          SUM1=0.0
          Z = 0.5
          DO 50 M =1,50,2
          S=M*3.1416
          B = 0.1
          CL = 3.0
          TAU3 = ((B/CL)**2)*TAU
          C1 =(1.0/M)*(EXP(-(S*S)*TAU3)*SIN(S*Z))
          SUM1 = SUM1+C1
C         ALL PRINT STATEMENTS ARE FOR VIEWING THE INTERMEDIATE
C         RESULTS ON SCREEN
C         PRINT*,SUM1
C         WRITE(10,*)SUM1
          THSL(M)=(4.0/3.1416)*SUM1
          THS=THSL(M)
   50     CONTINUE
          DO 11 I = 0,10
          R = I*0.1
          SUM=0.0
          DO 200 K = 1,50
          W = BM(K)*R
```

```
      U = BM(K)*BO
      C = EXP(-BM(K)*BM(K)*TAU)*(BESSJ0(W))/(U*BESSJ1(U))
C      WRITE(10,*)C,BM(K),BESSJ0(W),BESSJ1(U)
      SUM = SUM + C
C      PRINT*,SUM
      THIN(K) = 2.0*SUM
      TH1N=THIN(K)
  200 CONTINUE
C      PRINT *,'THIN =',R,THIN(K)
      THFC=TH1N*THS
C      WRITE(10,*)R,TH1N,THS,THFC
      WRITE(10,*)R,THFC
   11 CONTINUE
      STOP
      END
      FUNCTION BESSJ0(X)
C      RETURNS THE BESSEL FUNCTION J0(X) FOR ANY REAL X
      REAL*8 Y,P1,P2,P3,P4,P5,
     *       Q1,Q2,Q3,Q4,Q5,
     *       R1,R2,R3,R4,R5,R6,
     *       S1,S2,S3,S4,S5,S6
      DATA P1,P2,P3,P4,P5/
     *          1.D0,-.1098628627D-2,
     *       .2734510407D-4,
     *       -.2073370639D-5,.2093887211D-6/,
     *       Q1,Q2,Q3,Q4,Q5/-.1562499995D-1,
     *       .1430488765D-3,-.6911147651D-5,
     *       .7621095161D-6,-.934945152D-7/
      DATA R1,R2,R3,R4,R5,R6
     *         /57568490574.D0,
     *       -13362590354.D0,651619640.7D0,
     *          -11214424.18D0,77392.33017D0,-184.9052456D0/,
     *       S1,S2,S3,S4,S5,S6/57568490411.D0,1029532985.D0,
     *       9494680.718D0,59272.64853D0,267.8532712D0,1.D0/
      IF(ABS(X).LT.8.)THEN
      Y=X**2
      BESSJ0=(R1+Y*(R2+Y*(R3+Y*(R4+Y*(R5+Y*R6)))))
     *        /(S1+Y*(S2+Y*(S3+Y*(S4+Y*(S5+Y*S6)))))
      ELSE
      AX=ABS(X)
      Z=8./AX
      Y=Z**2
      XX=AX-.785398164
      BESSJ0=SQRT(.636619772/AX)
     * *(COS(XX)*(P1+Y*P2+Y*(P3+Y*(P4+Y
     * *P5))))-Z*SIN(XX)*(Q1+Y*(Q2+Y*(Q3+Y*(Q4+Y*Q5))))
      ENDIF
      RETURN
      END

      FUNCTION BESSJ1(X)
C      RETURNS THE BESSEL FUNCTION J1(X) FOR ANY REAL X
      REAL*8 Y,P1,P2,P3,P4,P5,
```

```
*   Q1,Q2,Q3,Q4,Q5,
*          R1,R2,R3,R4,R5,R6,
*   S1,S2,S3,S4,S5,S6
    DATA R1,R2,R3,R4,R5,R6/72362614232.D0,
*         -7895059235.D0,242396853.1D0,
*   -2972611.439D0,15704.48260D0,
*   -30.16036606D0/,

*   S1,S2,S3,S4,S5,S6/144725228442.D0,
*   2300535178.D0,
*   18583304.74D0,99447.43394D0,
*   376.9991397D0,1.D0/
    DATA P1,P2,P3,P4,P5/1.D0,.183105D-2,
*   -.3516396496D-4,.2457520174D-5,
*   -.240337019D-6/,Q1,Q2,Q3,Q4,Q5/
*   .04687499995D0,-.2002690873D-3,
*   .8449199096D-5,-.88228987D-6,.105787412D-6/
    IF(ABS(X).LT.8.)THEN
    Y=X**2
    BESSJ1=X*(R1+Y*(R2+Y*
*   (R3+Y*(R4+Y*(R5+Y*R6)))))
*   /(S1+Y*(S2+Y*(S3+Y*(S4+Y*(S5+Y*S6)))))
    ELSE
    AX=ABS(X)
    Z=8./AX
    Y=Z**2
    XX=AX-2.356194491
    BESSJ1=SQRT(.636619772/AX)
*   *(COS(XX)*(P1+Y*(P2+Y*(P3+Y*(P4+Y
*         *P5))))-Z*SIN(XX)*(Q1+Y*
*   (Q2+Y*(Q3+Y*(Q4+Y*Q5)))))
*   *SIGN(1.,X)
    ENDIF
    RETURN
    END
```

OUTPUT FOR TAU = 0.10

R	THFC
0.0000000E+00	0.8490568
0.1000000	0.8398013
0.2000000	0.8117947
0.3000000	0.7644722
0.4000000	0.6973760
0.5000000	0.6107559
0.6000000	0.5061774
0.7000000	0.3869560
0.8000000	0.2582564
0.9000000	0.1267717
1.000000	1.6550317E-05